XVII. *Researches in Physical Astronomy.* By J. W. LUBBOCK, *Esq. V.P. and Treas. R.S.*

Read June 7, 1832.

I SUBJOIN some further developments in the Theory of the Moon, which I have thought it advisable to give at length, in order to save the trouble of the calculator and to avoid the danger of mistake, although they may be obtained with great readiness and facility by means of the Table which I have given for the purpose.

While on the one hand it seems desirable to introduce into the science of Physical Astronomy a greater degree of uniformity, by bringing to perfection a Theory of the Moon, founded on the integration of the equations which are used in the planetary theory, it seems also no less important to complete in the latter the method hitherto applied solely to the periodic inequalities. Hitherto those terms in the disturbing function which give rise to the secular inequalities have been detached, and the stability of the system has been inferred by means of the integration of certain equations, which are linear when the higher powers of the eccentricities are neglected, and from considerations founded on the variation of the elliptic constants.

The stability of the system may, I think, also be inferred from the expressions which result at once from the direct integration of the differential equations. In fact, in order that the system may be stable, it is necessary that none of the angles under the sign *sine* or *cosine* be imaginary, which terms would then be converted into exponentials, and be subject to indefinite increase. In the lunar theory, the arbitrary quantities being determined with that view, according to the method here given, the angles which are introduced may be reduced to the difference of the mean motions of the sun and moon, their mean anomalies and the argument of the moon's latitude *.

* So that however far the approximation be carried, all the arguments, in the expressions of r, s, and λ are of the form, $it \pm kx \pm lz \pm my$; i, k, l, and m being some whole numbers.

This being the case, no imaginary angles are introduced, if the quantities c and g are rational. This theory, which does not seem to be limited by the direction of the moon's motion, and which may be extended without difficulty, already embraces the terms which are included in the secular inequalities, and which are derived from the constant part of R carried to the order of the squares of the eccentricities. Generally when the method of the variation of constants is employed to determine any inequalities, the development of R must be carried one degree further, as regards the eccentricities, than the degree which is required of the inequalities sought.

The equation for determining the coefficients of the expression for the reciprocal of the radius vector is,

$$\frac{d^2 . r^2}{2\,d\,t^2} - \frac{d^2 . r^3\,\delta \frac{1}{r}}{d\,t^2} + \frac{3\,d^2 . r^4 \left(\delta \frac{1}{r}\right)^2}{2\,d\,t^2} - \frac{2\,d^2 . r^5 \left(\delta \frac{1}{r}\right)^3}{d\,t^2} - \frac{\mu}{r} + \frac{\mu}{a} + 2\int d\,R + r\left(\frac{d\,R}{d\,r}\right) = 0$$

$$r^3\,\delta . \frac{1}{r} - \frac{3}{2}\left(r\,\delta \frac{1}{r}\right)^2 = \left\{\left\{1 + 3\,e^2\left(1 + \frac{e^2}{8}\right)\right\} r_1 - \frac{3\,e^2}{2}\left(1 + \frac{3}{8}\,e^2\right)(r_3 + r_4)\right.$$

$$\left. - \frac{3}{2}\left\{2\,r_0\,r_1 + e^2\,(r_3 + r_4)\,r_2 + e_i^2\,(r_6 + r_7)\,r_5\right\}\right\} \cos 2\,t + \&c.$$

$$- 3\,e^2\,\{2\,r_1\,r_2 + 2\,r_0\,r_3 + 2\,r_0\,r_4\}$$

r_n' being the coefficient corresponding to the n^{th} argument in the development of $r\,\delta \frac{1}{r}$. The development of $r^3\,\delta \frac{1}{r}$ is easily deduced from that of $r\,\delta \frac{1}{r}$ given in the Phil. Trans. 1832, Part I. p. 3, and that of $\left(r\,\delta \frac{1}{r}\right)^2$ from that of $\left(\delta \frac{1}{r}\right)^2$, p. 4. If $\mathfrak{r}_n$ is that part of the coefficient of the n^{th} argument in the development of the quantity $r^3\,\delta \frac{1}{r} - \frac{3}{2}\left(r\,\delta \frac{1}{r}\right)^2$ which is independent of r_n, with a contrary sign;

$$\mathfrak{r}_1 = \frac{3\,e^2}{2}\left(1 + \frac{3}{8}\,e^2\right)(r_3 + r_4) + \frac{3}{2}\left\{2\,r_0\,r_1 + e^2\,(r_3 + r_4)\,r_2 + e_i^2\,(r_6 + r_7)\,r_5\right\}$$

$$- 3\,e^2\,\{2\,r_1\,r_2 + 2\,r_0\,r_3 + 2\,r_0\,r_4\}$$

$$\mathfrak{r}_2 = \frac{3}{2}\left(1 + \frac{3}{8}\,e^2\right)(2\,r_0 + e^2\,r_8) + \frac{3}{2}\left\{(r_4 + r_3)\,r_1 + 2\,r_0\,r_2\right\}$$

$$- 6\left\{r_0^2 + \frac{r_1^2}{2} + \frac{e^2\,r_3^2}{2} + \&c.\right\}$$

$$\mathfrak{r}_3 = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right)(e^2 r_9 + r_1) + \frac{3}{2}\left\{r_1 r_2 + 2 r_0 r_3\right\}$$

$$- 3\left\{2 r_0 r_1 + e^2 (r_3 + r_4) r_2 + e_{\prime}^2 (r_6 + r_7) r_5\right\}$$

$$\mathfrak{r}_4 = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right)(r_1 + e^2 r_{10}) + \frac{3}{2}\left\{r_1 r_2 + 2 r_0 r_4\right\}$$

$$- 3\left\{2 r_0 r_1 + e^2 (r_3 + r_4) r_2 + e_{\prime}^2 (r_6 + r_7) r_5\right\}$$

$$\mathfrak{r}_5 = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right)(e^2 r_{14} + e^2 r_{11}) + \frac{3}{2}\left\{r_1 r_7 + r_1 r_6 + 2 r_0 r_5\right\}$$

$$\mathfrak{r}_6 = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right)(e^2 r_{12} + e^2 r_{16}) + \frac{3}{2}\left\{r_5 r_1 + 2 r_0 r_6\right\}$$

$$\mathfrak{r}_7 = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right)(e^2 r_{15} + e^2 r_{13}) + \frac{3}{2}\left\{r_5 r_1 + 2 r_0 r_7\right\}$$

$$\mathfrak{r}_8 = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right)(2 r_0 + r_2 + e^2 r_{20}) + \frac{e^2 r_2}{16} + \frac{3}{2}\left\{r_2^2 + r_4 r_3 + r_1 r_9 + r_1 r_{10}\right\}$$

$$- 3\left\{2 r_0^2 + r_1^2 + (r_4 + r_3) r_1 + 2 r_0 r_2\right\} + 3\left\{r_0^2 + \frac{r_1^2}{2}\right\}$$

$$\mathfrak{r}_9 = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right)(e^2 r_{21} + r_3) + \frac{e^2}{16} r_4 + \frac{3}{2}\left\{r_2 r_3 + 2 r_0 r_9\right\}$$

$$- 3\left\{r_1 r_2 + 2 r_0 r_3\right\} + \frac{3}{2}\left\{2 r_0 r_1 + e^2 r_3 r_2\right\}$$

$$\mathfrak{r}_{10} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right)(r_4 + e^2 r_{22}) + \frac{e^2}{16} r_3 + \frac{3}{2}\left\{r_4 r_2 + 2 r_0 r_{10}\right\}$$

$$- 3\left\{r_1 r_2 + 2 r_0 r_4\right\} + \frac{3}{2}\left\{2 r_0 r_1 + e^2 r_3 r_2\right\}$$

$$\mathfrak{r}_{11} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right)(r_5 + e^2 r_{23}) + \frac{3}{2}\left\{r_1 r_{13} + r_1 r_{12} + r_2 r_5 + r_6 r_4 + r_3 r_7 + 2 r_0 r_{11}\right\}$$

$$- 3\left\{r_1 r_7 + r_1 r_6 + 2 r_0 r_5\right\}$$

$$\mathfrak{r}_{12} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right)(e^2 r_{24} + r_6) + \frac{3}{2}\left\{r_{11} r_1 + r_2 r_6 + r_5 r_3 + 2 r_0 r_{12}\right\} - 3\left\{r_5 r_1 + 2 r_0 r_6\right\}$$

$$\mathfrak{r}_{13} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right)(r_7 + e^2 r_{25}) + \frac{3}{2}\left\{r_{11} r_1 + r_2 r_7 + r_5 r_4 + 2 r_0 r_{13}\right\} - 3\left\{r_5 r_1 + 2 r_0 r_7\right\}$$

$$\mathfrak{r}_{14} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right)(e^2 r_{26} + r_5) + \frac{3}{2}\left\{r_{16} r_1 + r_{15} r_1 + r_2 r_5 + r_6 r_3 + r_7 r_4 + 2 r_0 r_{14}\right\}$$

$$- 3\left\{r_1 r_7 + r_1 r_6 + 2 r_0 r_5\right\}$$

$$\mathfrak{r}_{15} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right)(e^2 r_{27} + r_7) + \frac{3}{2}\left\{r_{14} r_1 + r_2 r_7 + r_5 r_3\right\} - 3\left\{r_5 r_1 + 2 r_0 r_7\right\}$$

$$\mathfrak{r}_{16} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right)(r_6 + e^2 r_{28}) + \frac{3}{2}\left\{r_{14} r_1 + r_2 r_6 + r_5 r_4\right\} - 3\left\{r_5 r_1 + 2 r_0 r_6\right\}$$

$$\mathfrak{r}_{17} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right)(e^2 r_{32} + e^2 r_{29}) + \frac{3}{2}\left\{r_5{}^2 + r_7 r_6 + r_1 r_{18} + r_1 r_{19}\right\}$$

$$\mathfrak{r}_{18} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right)(e^2 r_{30} + e^2 r_{34}) + \frac{3}{2}\left\{r_{17} r_1 + r_5 r_6\right\}$$

$$\mathfrak{r}_{19} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right)(e^2 r_{33} + e^2 r_{31}) + \frac{3}{2}\left\{r_{17} r_1 + r_7 r_5\right\}$$

$$\mathfrak{r}_{20} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_8 + \frac{1}{8} r_0 \qquad \mathfrak{r}_{21} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_9 + \frac{1}{16} r_1$$

$$\mathfrak{r}_{22} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{10} + \frac{1}{16} r_1 \qquad \mathfrak{r}_{23} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{11}$$

$$\mathfrak{r}_{24} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{12} \qquad \mathfrak{r}_{25} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{13}$$

$$\mathfrak{r}_{26} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{14} \qquad \mathfrak{r}_{27} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{15} \qquad \mathfrak{r}_{28} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{16}$$

$$\mathfrak{r}_{29} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{17} \qquad \mathfrak{r}_{30} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{18} \qquad \mathfrak{r}_{31} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{19}$$

$$\mathfrak{r}_{32} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{17} \qquad \mathfrak{r}_{33} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{19} \qquad \mathfrak{r}_{34} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{18}$$

$$\mathfrak{r}_{35} = 0 \qquad \mathfrak{r}_{36} = 0 \qquad \mathfrak{r}_{37} = 0$$

$$\mathfrak{r}_{38} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{20} + \frac{1}{16} r_2 \qquad \mathfrak{r}_{39} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{21} + \frac{1}{16} r_3$$

$$\mathfrak{r}_{40} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{22} + \frac{1}{16} r_4 \qquad \mathfrak{r}_{41} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{23} + \frac{1}{16} r_5$$

$$\mathfrak{r}_{42} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{24} + \frac{1}{16} r_6 \qquad \mathfrak{r}_{43} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{25} + \frac{1}{16} r_7$$

$$\mathfrak{r}_{44} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{26} + \frac{1}{16} r_5 \qquad \mathfrak{r}_{45} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{27} + \frac{1}{16} r_7$$

$$\mathfrak{r}_{46} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{28} + \frac{1}{16} r_6 \qquad \mathfrak{r}_{47} = \frac{3}{2}\left(1 + \frac{3}{8}e^2\right) r_{29}$$

Let R_n be the coefficient corresponding to the n^{th} argument in the development of $a R + a \delta R$, $m R'_n$ the coefficient corresponding to the n^{th} argument in the development of $a \delta \, \mathrm{d} R$ with its sign changed, Phil. Trans. 1832, p. 161, so that, for example, when the square of the disturbing force is neglected,

$$R_1 = -\frac{3}{4}\frac{m_{\prime}}{\mu}\frac{a^3}{a_{\prime}{}^3} \quad \text{then}$$

$$r_1 \left\{1 + 3 e^2\left(1 + \frac{e^2}{8}\right)\right\} = \frac{(2-2m)^2}{(2-2m)^2-1}\mathfrak{r}_1 - \frac{2}{(2-2m)^2-1}\left\{\left\{\frac{2}{2-2m} + 1\right\} R_1 + \frac{m}{2-2m} R_1{}'\right\}$$

$$c^2\left\{1 - \frac{e^2}{8} - \mathfrak{r}_2\right\} = 1 - \frac{e^2}{8} - 2\left\{\left\{\frac{1}{c} + 1\right\} R_2 + \frac{m}{c} R_2{}'\right\}$$

$$r_3 \left\{1 + 3e^2\left(1 + \frac{e^2}{8}\right)\right\} = \frac{(2-2m-c)^2}{(2-2m-c)^2-1}\mathfrak{r}_3$$

$$- \frac{2}{(2-2m-c)^2-1}\left\{\left\{\frac{2-c}{2-2m-c} + 1\right\} R_3 + \frac{m}{2-2m-c} R_3'\right\}$$

$$r_4 \left\{1 + 3e^2\left(1 + \frac{e^2}{8}\right)\right\} = \frac{(2-2m+c)^2}{(2-2m+c)^2-1}\mathfrak{r}_4$$

$$- \frac{2}{(2-2m+c)^2-1}\left\{\left\{\frac{2+c}{2-2m+c} + 1\right\} R_4 + \frac{m}{2-2m+c} R_4'\right\}$$

$$r_5 \left\{1 + 3e^2\left(1 + \frac{e^2}{8}\right)\right\} = \frac{m^2}{m^2-1}\mathfrak{r}_5 - \frac{2}{m^2-1}\left\{R_5 + R_5'\right\}$$

$$r_6 \left\{1 + 3e^2\left(1 + \frac{e^2}{8}\right)\right\} = \frac{(2-3m)^2}{(2-3m)^2-1}\mathfrak{r}_6 - \frac{2}{(2-3m)^2-1}\left\{\left\{\frac{2}{2-3m} + 1\right\} R_6 + \frac{m}{2-3m} R_6'\right\}$$

$$r_7 \left\{1 + 3e^2\left(1 + \frac{e^2}{8}\right)\right\} = \frac{(2-m)^2}{(2-m)^2-1}\mathfrak{r}_7 - \frac{2}{(2-m)^2-1}\left\{\left\{\frac{2}{2-m} + 1\right\} R_7 + \frac{m}{2-m} R_7'\right\}$$

$$c^2 \left\{1 - \frac{e^2}{3} - 2r_8 - 2\mathfrak{r}_8\right\} = 1 - \frac{e^2}{3} - 2\left\{\left\{\frac{1}{c} + 1\right\} R_8 + \frac{m}{2c} R_8'\right\}$$

$$r_9 \left\{1 + 3e^2\left(1 + \frac{e^2}{8}\right)\right\} = \frac{(2-2m-2c)^2}{(2-2m-2c)^2-1}\mathfrak{r}_9$$

$$- \frac{2}{(2-2m-2c)^2-1}\left\{\left\{\frac{2-2c}{2-2m-2c} + 1\right\} R_9 + \frac{m}{2-2m-2c} R_9'\right\}$$

$$r_{10} \left\{1 + 3e^2\left(1 + \frac{e^2}{8}\right)\right\} = \frac{(2-2m+2c)^2}{(2-2m+2c)^2-1}\mathfrak{r}_{10}$$

$$- \frac{2}{(2-2m+2c)^2-1}\left\{\left\{\frac{2+2c}{2-2m+2c} + 1\right\} R_{10} + \frac{m}{2-2m+2c} R_{10}'\right\}$$

$$r_{11} \left\{1 + 3e^2\left(1 + \frac{e^2}{8}\right)\right\} = \frac{(c+m)^2}{(c+m)^2-1}\mathfrak{r}_{11} - \frac{2}{(c+m)^2-1}\left\{\left\{\frac{c}{c+m} + 1\right\} R_{11} + \frac{m}{c+m} R_{11}'\right\}$$

$$r_{12} \left\{1 + 3e^2\left(1 + \frac{e^2}{8}\right)\right\} = \frac{(2-c-3m)^2}{2-c-3m)^2-1}\mathfrak{r}_{12}$$

$$- \frac{2}{(2-c-3m)^2-1}\left\{\left\{\frac{2-c}{2-c-3m} + 1\right\} R_{12} + \frac{m}{2-c-3m} R_{12}'\right\}$$

$$r_{13} \left\{1 + 3e^2\left(1 + \frac{e^2}{8}\right)\right\} = \frac{(2-m+c)^2}{(2-m+c)^2-1}\mathfrak{r}_{13}$$

$$- \frac{2}{(2-m+c)^2-1}\left\{\left\{\frac{2+c}{2-m+c} + 1\right\} R_{13} + \frac{m}{2-m+c} R_{13}'\right\}$$

$$r_{14} \left\{1 + 3e^2\left(1 + \frac{e^2}{8}\right)\right\} = \frac{(c-m)^2}{(c-m)^2-1}\mathfrak{r}_{14} - \frac{2}{(c-m)^2-1}\left\{\left\{\frac{c}{c-m} + 1\right\} R_{14} + \frac{m}{c-m} R_{14}'\right\}$$

$$r_{15}\left\{1+3e^2\left(1+\frac{e^2}{8}\right)\right\}=\frac{(2-m-c)^2}{(2-m-c)^2-1}\mathfrak{r}_{15}$$

$$-\frac{2}{(2-m-c^2)-1}\left\{\left\{\frac{2-c}{2-m-c}+1\right\}R_{15}+\frac{m}{2-m-c}R_{15}'\right\}$$

$$r_{16}\left\{1+3e^2\left(1+\frac{e^2}{8}\right)\right\}=\frac{(2-3m+c)^2}{(2-3m+c)^2-1}\mathfrak{r}_{16}$$

$$-\frac{2}{(2-3m+c)^2-1}\left\{\left\{\frac{2+c}{2-3m+c}+1\right\}R_{16}+\frac{m}{2-3m+c}R_{16}'\right\}$$

$$r_{17}\left\{1+3e^2\left(1+\frac{e^2}{8}\right)\right\}=\frac{4m^2}{4m^2-1}\mathfrak{r}_{17}-\frac{2}{4m^2-1}\left\{R_{17}+R'_{17}\right\}$$

$$r_{18}\left\{1+3e^2\left(1+\frac{e^2}{8}\right)\right\}=\frac{(2-4m)^2}{(2-4m)^2-1}\mathfrak{r}_{18}$$

$$-\frac{2}{(2-4m)^2-1}\left\{\left\{\frac{2}{2-4m}+1\right\}R_{18}+\frac{m'}{2-4m}R'_{18}\right\}$$

$$r_{19}\left\{1+3e^2\left(1+\frac{e^2}{8}\right)\right\}=\frac{4}{3}\mathfrak{r}_{19}-\frac{2}{3}\left\{2R_{19}+\frac{m}{2}R'_{19}\right\}$$

$$r_{101}\left\{1+3e^2\left(1+\frac{e^2}{8}\right)\right\}=\frac{(1-m)^2}{(1-m)^2-1}\mathfrak{r}_{101}$$

$$-\frac{1}{(1-m)^2-1}\left\{\left\{\frac{2}{1-m}+3\right\}R_{101}+\frac{2m}{1-m}R_{101}'\right\}$$

$$r_{102}\left\{1+3e^2\left(1+\frac{e^2}{8}\right)\right\}=\frac{(1-m-c)^2}{(1-m-c)^2-1}\mathfrak{r}_{102}$$

$$-\frac{1}{(1-m-c)^2-1}\left\{\left\{\frac{2(1-c)}{1-m-c}+3\right\}R_{102}+\frac{2m}{1-m-c}R_{102}'\right\}$$

$$r_{103}\left\{1+3e^2\left(1+\frac{e^2}{8}\right)\right\}=\frac{(1-m+c)^2}{(1-m+c)^2-1}\mathfrak{r}_{103}$$

$$-\frac{1}{(1-m+c)^2-1}\left\{\left\{\frac{2(1+c)}{(1-m+c)}+3\right\}R_{103}+\frac{2m}{1-m+c}R_{103}'\right\}$$

$$r_{104}\left\{1+3e^2\left(1+\frac{e^2}{8}\right)\right\}=\frac{(1-2m)^2}{(1-2m)^2-1}\mathfrak{r}_{104}$$

$$-\frac{1}{(1-2m)^2-1}\left\{\left\{\frac{2}{1-2m}+3\right\}R_{104}+\frac{2m}{1-2m}R_{104}'\right\}$$

$$m=\cdot 0748013 \qquad c=\cdot 991548 \qquad e=\cdot 0548442$$

Substituting in the preceding equations, and writing the logarithms of the coefficients instead of the coefficients themselves, we get

$$r_1=0{\cdot}1460995\,\mathfrak{r}_1-0{\cdot}2308405\,R_1-8{\cdot}5192440\,R_1'$$

$$r_3=-0{\cdot}4450058\,\mathfrak{r}_3+1{\cdot}2154967\,R_3+9{\cdot}8181930\,R_3'$$

$$r_4 = 0{\cdot}0535010\,\mathfrak{r}_4 - 9{\cdot}7596140\,R_4 - 7{\cdot}8675954\,R_4'$$

$$r_5 = -\,7{\cdot}7463524\,\mathfrak{r}_5 + 0{\cdot}2995642\,R_5 + 0{\cdot}2995642\,R_5'$$

$$r_6 = 0{\cdot}1617938\,\mathfrak{r}_6 - 0{\cdot}2917755\,R_6 - 8{\cdot}5887003\,R'_6$$

$$r_7 = 0{\cdot}1326574\,\mathfrak{r}_7 - 0{\cdot}1741219\,R_7 - 8{\cdot}4541703\,R_7'$$

$$r_9 = -\,8{\cdot}2495414\,\mathfrak{r}_9 + 0{\cdot}2456727\,R_9 - 0{\cdot}0558873\,R_9'$$

$$r_{10} = 0{\cdot}0267023\,\mathfrak{r}_{10} - 9{\cdot}4699640\,R_{10} - 7{\cdot}4508570\,R_{10}'$$

$$r_{11} = 0{\cdot}9148582\,\mathfrak{r}_{11} - 1{\cdot}4456131\,R_{11} - 0{\cdot}0060992\,R_{11}'$$

$$r_{12} = 0{\cdot}1990183\,\mathfrak{r}_{12} + 1{\cdot}0704790\,R_{12} + 9{\cdot}6909293\,R_{12}'$$

$$r_{13} = 0{\cdot}0504044\,\mathfrak{r}_{13} - 9{\cdot}7282013\,R_{13} - 7{\cdot}8306471\,R_{13}'$$

$$r_{14} = 0{\cdot}7176313\,\mathfrak{r}_{14} + 1{\cdot}4125573\,R_{14} + 0{\cdot}0058216\,R_{14}'$$

$$r_{15} = -\,0{\cdot}8282531\,\mathfrak{r}_{15} + 1{\cdot}5070002\,R_{15} + 0{\cdot}0926384\,R_{15}'$$

$$r_{16} = 0{\cdot}0568761\,\mathfrak{r}_{16} - 9{\cdot}7921334\,R_{16} - 7{\cdot}9057198\,R_{16}'$$

$$r_{17} = -\,8{\cdot}3558051\,\mathfrak{r}_{17} + 0{\cdot}3069571\,R_{17} + 0{\cdot}3069571\,R_{17}'$$

$$r_{18} = 0{\cdot}1803182\,\mathfrak{r}_{18} - 0{\cdot}3576881\,R_{18} - 8{\cdot}6633026\,R_{18}'$$

$$r_{19} = 0{\cdot}1210357\,\mathfrak{r}_{19} - 0{\cdot}1210357\,R_{19} - 8{\cdot}3928848\,R_{19}'$$

$$r_{101} = -\,0{\cdot}7701834\,\mathfrak{r}_{101} + 1{\cdot}5505062\,R_{101} + 0{\cdot}0464175\,R_{101}'$$

$$r_{102} = -\,7{\cdot}6416818\,\mathfrak{r}_{102} + 0{\cdot}4365911\,R_{102} - 0{\cdot}3511177\,R_{102}'$$

$$r_{103} = 0{\cdot}1340779\,\mathfrak{r}_{103} - 0{\cdot}2746455\,R_{103} - 8{\cdot}4613229\,R_{103}'$$

$$r_{104} = -\,0{\cdot}4131392\,\mathfrak{r}_{104} + 1{\cdot}2823979\,R_{104} + 9{\cdot}7992116\,R_{104}'$$

These quantities introduce into the expression for the longitude expressed in sexagesimal seconds, the terms,

$$+\{5{\cdot}4942896\,\mathfrak{r}_1 - 5{\cdot}5790306\,R_1 - 3{\cdot}8674341\,R_1'\}\sin 2t \qquad [4{\cdot}7798951]$$

$$+\{-\,4{\cdot}8656743\,\mathfrak{r}_3 + 5{\cdot}6361652\,R_3 + 4{\cdot}2382615\,R_3'\}\sin(2t - x) \qquad [4{\cdot}1857212]$$

$$+\{3{\cdot}9544710\,\mathfrak{r}_4 - 3{\cdot}6605840\,R_4 - 1{\cdot}7685654\,R_4'\}\sin(2t + x) \qquad [3{\cdot}1463242]$$

$$+\{-\,2{\cdot}7130189\,\mathfrak{r}_5 + 5{\cdot}2662307\,R_5 + 5{\cdot}2662307\,R_5'\}\sin z \qquad [5{\cdot}7917274]$$

$$+\{3{\cdot}7530252\,\mathfrak{r}_6 - 3{\cdot}8830069\,R_6 - 2{\cdot}1799317\,R_6'\}\sin(2t - z) \qquad [3{\cdot}0408572]$$

$$+\{3{\cdot}6887576\,\mathfrak{r}_7 - 3{\cdot}7302221\,R_7 - 2{\cdot}0102705\,R_7'\}\sin(2t + z) \qquad [2{\cdot}9705948]$$

$$+\{2{\cdot}2203935\,\mathfrak{r}_9 - 4{\cdot}2165248\,R_9 + 4{\cdot}0267394\,R_9'\}\sin(2t - 2x) \qquad [4{\cdot}5469577]$$

$$+\{2{\cdot}5368240\,\mathfrak{r}_{10} - 1{\cdot}9800857\,R_{10} - 9{\cdot}9609787\,R_{10}'\}\sin(2t + 2x) \qquad [1{\cdot}6254969]$$

$$+\{3{\cdot}4666708\,\mathfrak{r}_{11} - 3{\cdot}9974257\,R_{11} - 2{\cdot}5579118\,R_{11}'\}\sin(x + z) \qquad [2{\cdot}2228889]$$

$+\{-2{\cdot}8843819\,\mathfrak{r}_{12}+3{\cdot}7558426\,R_{12}+2{\cdot}3762929\,R_{12}'\}\sin(2t-x-z)$ [2·4899904]

$+\{2{\cdot}1652119\,\mathfrak{r}_{13}-1{\cdot}8430088\,R_{13}-9{\cdot}9454546\,R_{13}'\}\sin(2t+x+z)$ [1·3488787]

$+\{-3{\cdot}3350850\,\mathfrak{r}_{14}+4{\cdot}0300110\,R_{14}+2{\cdot}6232753\,R_{14}'\}\sin(x-z)$ [2·3541741]

$+\{-3{\cdot}4377718\,\mathfrak{r}_{15}+4{\cdot}1169189\,R_{15}+2{\cdot}7021571\,R_{15}'\}\sin(2t-x+z)$ [2·3383041]

$+\{2{\cdot}1945476\,\mathfrak{r}_{16}-1{\cdot}9298049\,R_{16}-0{\cdot}0433913\,R_{16}'\}\sin(2t+x-z)$ [1·3946097]

$+\{-1{\cdot}2465621\,\mathfrak{r}_{17}+3{\cdot}1977141\,R_{17}+3{\cdot}1977141\,R_{17}'\}\sin 2z$ [3·4147879]

$+\{2{\cdot}0153626\,\mathfrak{r}_{18}-2{\cdot}1927325\,R_{18}-0{\cdot}4983470\,R_{18}'\}\sin(2t-2z)$ [1·3033627]

$+\{1{\cdot}8857018\,\mathfrak{r}_{19}-1{\cdot}8857018\,R_{19}-0{\cdot}1575509\,R_{19}'\}\sin(2t+2z)$ [1·1626061]

$+\{-6{\cdot}4194035\,\mathfrak{r}_{101}+7{\cdot}1997263\,R_{101}+5{\cdot}6956376\,R_{101}'\}\sin t$ [5·3481901]

$+\{3{\cdot}1744332\,\mathfrak{r}_{102}-5{\cdot}9693425\,R_{102}+5{\cdot}8838691\,R_{102}'\}\sin(t-x)$ [6·4098870]

$+\{4{\cdot}2060990\,\mathfrak{r}_{103}-4{\cdot}3466666\,R_{103}-2{\cdot}5333440\,R_{103}'\}\sin(t+x)$ [3·4884264]

$+\{-4{\cdot}3240929\,\mathfrak{r}_{104}+5{\cdot}1933516\,R_{104}+3{\cdot}7101653\,R_{104}'\}\sin(t-z)$ [3·6803018]

The preceding expressions serve to show the extent to which the approximation must be carried in the calculation of the quantities $\mathfrak{r}$, R, &c.

If we take the term $5{\cdot}6361652\,R_3$, since log. $\frac{m_{,}a^3}{\mu\,a_{,}^3}=7{\cdot}7464329$, it is evident that in order not to neglect ·01″ in the value of λ, the coefficient of $\frac{m_{,}a^2}{\mu\,a_{,}^3}\cos(2t-x)$ in the development of δR must be calculated exactly to the fifth place of decimals, but not beyond. The number 4·1857212 is the logarithm of the quantity $\frac{e}{(2-m-c)^2}$, expressed in sexagesimal seconds, and serves to show in like manner how far the approximation must be carried in the calculation of $\frac{\mathrm{d}R}{\mathrm{d}\lambda}$.

When the square of the disturbing force is neglected,

$$R_2=\frac{m_{,}a^3}{2\mu\,a_{,}^3}\qquad R_8=\frac{m_{,}a^3}{8\mu\,a_{,}^3}\qquad r_0=-\frac{m_{,}a^3}{2\mu\,a_{,}^3}\qquad \mathfrak{r}_2=3\,r_0\qquad \mathfrak{r}_8=3\,r_0\qquad r_2=0$$

$$c^2=1+3\,r_0-\frac{2\,m_{,}a^3}{\mu\,a_{,}^3}=1-\frac{7\,m_{,}a^3}{2\mu\,a_{,}^3}$$

The equation of p. 5, line 8, gives $r_8=0$.

$$\frac{\mathrm{d}\lambda'}{\mathrm{d}t}=\frac{h}{r'^2}\left\{1-\frac{1}{h}\int\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\mathrm{d}t+\frac{1}{2h^2}\left\{\int\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\mathrm{d}t\right\}^2\right.$$

$$\frac{h}{r'^2}=\frac{h(1+s^2)}{r^2}=\frac{h}{a^2}\left\{\frac{a}{\mathfrak{r}}+a\,\delta\frac{1}{r}\right\}^2\left\{1+s^2\right\}$$

$$= \frac{h}{a^2}\left\{\frac{a^2}{\mathrm{r}^2} + \frac{2a^2}{\mathrm{r}}\,\delta\frac{1}{r} + a^2\left(\delta.\frac{1}{r}\right)^2\right\}\left\{1 + s^2\right\}$$

$$s = \underset{[146]}{\gamma \sin y} + \underset{[147]}{\gamma s_{147} \sin(2t - y)} \text{ nearly}$$

$$s^2 = \frac{\gamma^2}{2} + \frac{\gamma^2 s^2_{147}}{2} - \underset{(1)}{\gamma^2 s_{147}\cos 2t} - \underset{(62)}{\frac{\gamma^2}{2}\cos 2y} + \underset{(63)}{\gamma^2 s_{147}\cos(2t - 2y)}$$

$$1 + s^2 = 1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2}s^2_{147}\left\{1 - \underset{[1]}{\gamma^2 s_{147}\cos 2t} - \underset{[62]}{\frac{\gamma^2}{2}\cos 2y} + \underset{[63]}{\gamma^2 s^2_{147}\cos(2t - 2y)}\right\}$$

nearly

$$\frac{a^2}{\mathrm{r}^2} = 1 + \frac{e^2}{2}\left(1 + \frac{3}{4}e^2\right) + \underset{[2]}{2e\left(1 + \frac{3}{8}e^2\right)\cos x} + \underset{[8]}{\frac{5}{2}e^2\left(1 + \frac{2}{15}e^2\right)\cos 2x}$$

$$+ \underset{[20]}{\frac{13}{4}e^3\cos 3x} + \underset{[38]}{\frac{103}{24}e^4\cos 4x}$$

$$\frac{a}{\mathrm{r}} = 1 + \underset{[2]}{e\left(1 - \frac{e^2}{8}\right)\cos x} + \underset{[8]}{e^2\left(1 - \frac{e^2}{3}\right)\cos 2x} + \underset{[20]}{\frac{9}{8}e^3\cos 3x} + \underset{[38]}{\frac{4}{3}e^4\cos 4x}$$

If the coefficients corresponding to the different arguments in the quantity $\frac{a^2}{r^2}$, be called $2\,\mathfrak{r}'_n$ and the coefficients of the different arguments in the development of the quantity

$$-na\left\{\int\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\,\mathrm{d}t - \frac{1}{2h^2}\left\{\int\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\,\mathrm{d}t^2\right\}^2\right\} \text{ be called } \mathfrak{R}_n\text{, then}$$

$$2\,\mathfrak{r}'_0 = \left\{1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2}s^2_{147}{}^*\right\}\left\{1 + \frac{e^2}{2}\left(1 + \frac{3}{4}e^2\right) + 2r_0 + r_0^2 + \frac{r_1^2}{2} + \frac{e^2 r_2^2}{2} + \frac{e^2 r_3^2}{2}\,\frac{e^2 r_4^2}{2} + \frac{e_{\prime}^2 r_5^2}{2}\right.$$

$$\left. + \frac{e_{\prime}^2 r_6^2}{2} + \frac{e_{\prime}^2 r_7^2}{2}\right\}$$

$$\mathfrak{r}'_1 = \left\{1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2}s^2_{147}\right\}\left\{r_1 - \gamma^2 s_{147} + \frac{e^2}{2}\left(1 - \frac{e^2}{8}\right)\left\{r_3 + r_4\right\} + \frac{e^4}{2}\left\{r_9 + r_{10}\right\} + 2r_0 r_1\right.$$

$$\left. + e^2(r_3 + r_4)\,r_2 + e_{\prime}^2(r_6 + r_7)\,r_5\right\}$$

$$\mathfrak{r}'_2 = \left\{1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2}s^2_{147}\right\}\left\{1 + \frac{3}{8}e^2 + r_2 + \frac{1}{2}\left(1 - \frac{e^2}{8}\right)\left\{2r_0 + e^2 r_8\right\} + \frac{e^2}{2}r_2\right.$$

$$\left. + (r_4 + r_3)\,r_1 + 2r_0 r_2\right\}$$

* $(s_{147})^2$ is intended.

$$\text{r}_3' = \left\{1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s^2_{147}\right\} \left\{r_3 + \frac{1}{2}\left(1 - \frac{e^2}{8}\right)\left\{e^2 r_9 + r_1 - \gamma^2 s_{147}\right\} + \frac{e^2}{2} r_4 + r_1 r_2 + 2 r_0 r_3\right\}$$

$$\text{r}_4' = \left\{1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s^2_{147}\right\} \left\{r_4 + \frac{1}{2}\left(1 - \frac{e^2}{8}\right)\left\{r_1 - \gamma^2 s_{147} + e^2 r_{10}\right\} + \frac{e^2}{2} r_3 + r_1 r_2 + 2 r_0 r_4\right\}$$

$$\text{r}_5' = \left\{1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s^2_{147}\right\} \left\{r_5 + \frac{1}{2}\left(1 - \frac{e^2}{8}\right)\left\{e^2 r_{14} + e^2 r_{11}\right\} + r_1 r_7 + r_1 r_6 + 2 r_0 r_5\right\}$$

$$\text{r}_6' = \left\{1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s^2_{147}\right\} \left\{r_6 + \frac{1}{2}\left(1 - \frac{e^2}{8}\right)\left\{e^2 r_{12} + e^2 r_{16}\right\} + r_5 r_1 + 2 r_0 r_6\right\}$$

$$\text{r}_7' = \left\{1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s^2_{147}\right\} \left\{r_7 + \frac{1}{2}\left(1 - \frac{e^2}{8}\right)\left\{e^2 r_{15} + e^2 r_{13}\right\} + r_5 r_1 + 2 r_0 r_7\right\}$$

$$\text{r}_8' = \left\{1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s^2_{147}\right\} \left\{r_8 + \frac{1}{2}\left(1 - \frac{e^2}{8}\right)\left\{r_2 + e^2 r_{20}\right\} + r_0 + \frac{9}{16} e^2 r_2\right.$$

$$\left. + r_2^2 + r_4 r_3 + r_1 r_9 + r_1 r_{10}\right\}$$

$$\text{r}_9' = \left\{1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s^2_{147}\right\} \left\{r_9 + \frac{1}{2}\left(1 - \frac{e^2}{8}\right)\left\{e^2 r_{21} + r_3\right\} + \frac{r_1}{2} - \frac{\gamma^2}{2} s_{147}\right.$$

$$\left. + \frac{9}{16} e^2 r_4 + r_2 r_3 + 2 r_0 r_9\right\}$$

$$\text{r}_{10}' = \left\{1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s^2_{147}\right\} \left\{r_{10} + \frac{1}{2}\left(1 - \frac{e^2}{8}\right)\left\{r_4 + e^2 r_{22}\right\} + \frac{r_1}{2} - \frac{\gamma^2}{2} s_{147}\right.$$

$$\left. + \frac{9}{16} e^2 r_3 + r_4 r_2 + 2 r_0 r_{10}\right\}$$

$$\text{r}_{11}' = \left\{1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s^2_{147}\right\} \left\{r_{11} + \frac{1}{2}\left(1 - \frac{e^2}{8}\right)\left\{r_5 + e^2 r_{23}\right\} + \frac{e^2}{2} r_{14} + r_1 r_{13} + r_1 r_{12} + r_2 r_5\right.$$

$$\left. + r_6 r_4 + r_3 r_7 + 2 r_0 r_{11}\right\}$$

$$\text{r}_{12}' = \left\{1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s^2_{147}\right\} \left\{r_{12} + \frac{1}{2}\left(1 - \frac{e^2}{8}\right)\left\{e^2 r_{24} + r_6\right\} + \frac{e^2}{2} r_{16} + r_{11} r_1 + r_2 r_6 + r_5 r_3 + 2 r_0 r_{12}\right\}$$

$$\text{r}_{13}' = \left\{1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s^2_{147}\right\} \left\{r_{13} + \frac{1}{2}\left(1 - \frac{e^2}{8}\right)\left\{r_7 + e^2 r_{25}\right\} + \frac{e^2}{2} r_{15} + r_{11} r_1 + r_2 r_7 + r_5 r_4 + 2 r_0 r_{13}\right\}$$

$$\text{r}_{14}' = \left\{1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s^2_{147}\right\} \left\{r_{14} + \frac{1}{2}\left(1 - \frac{e^2}{8}\right)\left\{e^2 r_{26} + r_5\right\} + \frac{e^2}{2} r_{11} + r_{16} r_1 + r_{15} r_1 + r_2 r_5 + r_6 r_3\right.$$

$$\left. + r_7 r_4 + 2 r_0 r_{14}\right\}$$

$$\mathfrak{r}_{15}^{\backprime} = \left\{1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s^2{}_{147}\right\} \left\{r_{15} + \frac{1}{2}\left(1 - \frac{e^2}{8}\right)\left\{e^2 r_{27} + r_7\right\} + \frac{e^2}{2} r_{13} + r_{14} r_1 + r_2 r_7 + r_5 r_3\right\}$$

$$\mathfrak{r}_{16}^{\backprime} = \left\{1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s^2{}_{147}\right\} \left\{r_{16} + \frac{1}{2}\left(1 - \frac{e^2}{8}\right)\left\{r_6 + e^2 r_{28}\right\} + \frac{e^2}{2} r_{12} + r_{14} r_1 + r_2 r_6 + r_5 r_4\right\}$$

$$\mathfrak{r}_{17}^{\backprime} = \left\{1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s^2{}_{147}\right\} \left\{r_{17} + \frac{1}{2}\left(1 - \frac{e^2}{8}\right)\left\{e^2 r_{32} + e^2 r_{29}\right\} + r_5^2 + r_7 r_6 + r_1 r_{18} + r_1 r_{19}\right\}$$

$$\mathfrak{r}_{18}^{\backprime} = \left\{1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s^2{}_{147}\right\} \left\{r_{18} + \frac{1}{2}\left(1 - \frac{e^2}{8}\right)\left\{e^2 r_{30} + e^2 r_{34}\right\} + r_{17} r_1 + r_5 r_6\right\}$$

$$\mathfrak{r}_{19}^{\backprime} = \left\{1 + \frac{\gamma^2}{2} + \frac{\gamma^2}{2} s^2{}_{147}\right\} \left\{r_{19} + \frac{1}{2}\left(1 - \frac{e^2}{8}\right)\left\{e^2 r_{33} + e^2 r_{31}\right\} + r_{17} r_1 + r_7 r_5\right\}$$

$$\lambda^{\backprime} = \left\{\frac{2h\mathfrak{r}_0^{\backprime}}{a^2} + 2\mathfrak{r}_0^{\backprime}\mathfrak{R}_0 + \mathfrak{r}_1^{\backprime}\mathfrak{R}_1 + e^2\mathfrak{r}_3^{\backprime}\mathfrak{R}_3 + e^2\mathfrak{r}_4^{\backprime}\mathfrak{R}_4 + e_{,}^2\mathfrak{r}_5^{\backprime}\mathfrak{R}_5 + e_{,}^2\mathfrak{r}_6^{\backprime}\mathfrak{R}_6 + e_{,}^2\mathfrak{r}_7^{\backprime}\mathfrak{R}_7\right\} t$$

$$+ \frac{1}{2-2m}\{2\mathfrak{r}_1^{\backprime} + 2\mathfrak{r}_0^{\backprime}\mathfrak{R}_1 + 2\mathfrak{r}_1^{\backprime}\mathfrak{R}_0 + e^2\mathfrak{r}_2^{\backprime}\mathfrak{R}_3 + e^2\mathfrak{r}_2^{\backprime}\mathfrak{R}_4 + e^2\mathfrak{r}_3^{\backprime}\mathfrak{R}_2 + e^2\mathfrak{r}_4\mathfrak{R}_2$$
$$+ e_{,}^2\mathfrak{r}_5^{\backprime}\mathfrak{R}_6 + e_{,}^2\mathfrak{r}_5^{\backprime}\mathfrak{R}_7 + e_{,}^2\mathfrak{r}_6^{\backprime}\mathfrak{R}_5 + e_{,}^2\mathfrak{r}_7^{\backprime}\mathfrak{R}_5\}\sin 2t$$

[1]

$$+ \frac{1}{c}\{2\mathfrak{r}_2^{\backprime} + 2\mathfrak{r}_0^{\backprime}\mathfrak{R}_2 + 2\mathfrak{r}_2^{\backprime}\mathfrak{R}_0 + \mathfrak{r}_1^{\backprime}\mathfrak{R}_4 + \mathfrak{r}_1^{\backprime}\mathfrak{R}_3 + \mathfrak{r}_3^{\backprime}\mathfrak{R}_1 + \mathfrak{r}_4^{\backprime}\mathfrak{R}_1\} e\sin x$$

[2]

$$+ \frac{1}{(2-2m-c)}\{2\mathfrak{r}_3^{\backprime} + 2\mathfrak{r}_0^{\backprime}\mathfrak{R}_3 + 2\mathfrak{r}_3^{\backprime}\mathfrak{R}_0 + \mathfrak{r}_1^{\backprime}\mathfrak{R}_2 + \mathfrak{r}_2^{\backprime}\mathfrak{R}_1\} e\sin(2t - x)$$

(3)

$$+ \frac{1}{(2-2m+c)}\{2\mathfrak{r}_4^{\backprime} + 2\mathfrak{r}_0^{\backprime}\mathfrak{R}_4 + 2\mathfrak{r}_4^{\backprime}\mathfrak{R}_0 + \mathfrak{r}_1^{\backprime}\mathfrak{R}_2 + \mathfrak{r}_2^{\backprime}\mathfrak{R}_1\} e\sin(2t + x)$$

[4]

$$+ \frac{1}{m}\{2\mathfrak{r}_5^{\backprime} + 2\mathfrak{r}_0^{\backprime}\mathfrak{R}_5 + 2\mathfrak{r}_5^{\backprime}\mathfrak{R}_0 + \mathfrak{r}_1^{\backprime}\mathfrak{R}_7 + \mathfrak{r}_1^{\backprime}\mathfrak{R}_6 + \mathfrak{r}_6^{\backprime}\mathfrak{R}_1 + \mathfrak{r}_7^{\backprime}\mathfrak{R}_1\} e_{,}\sin z$$

[5]

$$+ \frac{1}{(2-3m)}\{2\mathfrak{r}_6^{\backprime} + 2\mathfrak{r}_0\mathfrak{R}_6 + 2\mathfrak{r}_6^{\backprime}\mathfrak{R}_0 + \mathfrak{r}_1\mathfrak{R}_5 + \mathfrak{r}_5^{\backprime}\mathfrak{R}_1\} e_{,}\sin(2t - z)$$

[6]

$$+ \frac{1}{(2-m)}\{2\mathfrak{r}_7^{\backprime} + 2\mathfrak{r}_0^{\backprime}\mathfrak{R}_7 + 2\mathfrak{r}_7^{\backprime}\mathfrak{R}_0 + \mathfrak{r}_1^{\backprime}\mathfrak{R}_5 + \mathfrak{r}_5^{\backprime}\mathfrak{R}_1\} e_{,}\sin(2t + z)$$

[7]

$$+ \frac{1}{2c}\{2\mathfrak{r}_8^{\backprime} + 2\mathfrak{r}_0\mathfrak{R}_8 + 2\mathfrak{r}_8^{\backprime}\mathfrak{R}_0 + \mathfrak{r}_1^{\backprime}\mathfrak{R}_{10} + \mathfrak{r}_1^{\backprime}\mathfrak{R}_9 + \mathfrak{r}_2^{\backprime}\mathfrak{R}_2 + \mathfrak{r}_3^{\backprime}\mathfrak{R}_4 + \mathfrak{r}_4^{\backprime}\mathfrak{R}_3 + \mathfrak{r}_9^{\backprime}\mathfrak{R}_1 + \mathfrak{r}_{10}^{\backprime}\mathfrak{R}_1\} e^2\sin 2x$$

[8]

$$+ \frac{1}{(2-2m-2c)}\{2\mathfrak{r}_9^{\backprime} + 2\mathfrak{r}_0\mathfrak{R}_9 + 2\mathfrak{r}_9^{\backprime}\mathfrak{R}_0 + \mathfrak{r}_1^{\backprime}\mathfrak{R}_8 + \mathfrak{r}_2^{\backprime}\mathfrak{R}_3 + \mathfrak{r}_3'\mathfrak{R}_2 + \mathfrak{r}_8^{\backprime}\mathfrak{R}_1\} e^2\sin(2t - 2x)$$

[9]

$$+\frac{1}{(2-2m+2c)}\left\{2\,\mathfrak{r}_{10}{}' + 2\,\mathfrak{r}_0\,\mathfrak{R}_{10} + 2\,\mathfrak{r}_{10}{}'\mathfrak{R}_0 + \mathfrak{r}_1{}'\mathfrak{R}_8 + \mathfrak{r}_2{}'\mathfrak{R}_4 + \mathfrak{r}_4{}'\mathfrak{R}_2 + \mathfrak{r}_8{}'\mathfrak{R}_1\right\} e^2 \sin(2t+2z) \qquad [10]$$

$$+\frac{1}{(c+m)}\left\{2\,\mathfrak{r}_{11}{}' + 2\,\mathfrak{r}_0{}'\mathfrak{R}_{11} + 2\,\mathfrak{r}_{11}{}'\mathfrak{R}_0 + \mathfrak{r}_1{}'\mathfrak{R}_{13} + \mathfrak{r}_1{}'\mathfrak{R}_{12} + \mathfrak{r}_2{}'\mathfrak{R}_5 + \mathfrak{r}_3{}'\mathfrak{R}_7 + \mathfrak{r}_4{}'\mathfrak{R}_6 + \mathfrak{r}_5{}'\mathfrak{R}_2\right.$$
$$\left. + \mathfrak{r}_6{}'\mathfrak{R}_4 + \mathfrak{r}_7{}'\mathfrak{R}_3 + \mathfrak{r}_{12}{}'\mathfrak{R}_1 + \mathfrak{r}_{13}{}'\mathfrak{R}_1\right\} e\,e_{\prime} \sin(x+z) \qquad [11]$$

$$+\frac{1}{(2-3m-c)}\left\{2\,\mathfrak{r}_{12}{}' + 2\,\mathfrak{r}_0\,\mathfrak{R}_{12} + 2\,\mathfrak{r}_{12}{}'\mathfrak{R}_0 + \mathfrak{r}_1{}'\mathfrak{R}_{11} + \mathfrak{r}_2{}'\mathfrak{R}_6 + \mathfrak{r}_3{}'\mathfrak{R}_5 + \mathfrak{r}_5{}'\mathfrak{R}_3 + \mathfrak{r}_6{}'\mathfrak{R}_2\right.$$
$$\left. + \mathfrak{r}_{11}{}'\mathfrak{R}_1\right\} e\,e_{\prime} \sin(2t-x-z) \qquad [12]$$

$$+\frac{1}{(2-m+c)}\left\{2\,\mathfrak{r}_{13}{}' + 2\,\mathfrak{r}_0{}'\mathfrak{R}_{13} + 2\,\mathfrak{r}_{13}{}'\mathfrak{R}_0 + \mathfrak{r}_1\,\mathfrak{R}_{11} + \mathfrak{r}_2{}'\mathfrak{R}_7 + \mathfrak{r}_4{}'\mathfrak{R}_5 + \mathfrak{r}_5{}'\mathfrak{R}_4\right.$$
$$\left. + \mathfrak{r}_7{}'\mathfrak{R}_2 + \mathfrak{r}_{11}{}'\mathfrak{R}_1\right\} e\,e_{\prime} \sin(2t+x+z) \qquad [13]$$

$$+\frac{1}{(c-m)}\left\{2\,\mathfrak{r}_{14}{}' + 2\,\mathfrak{r}_0{}'\mathfrak{R}_{14} + 2\,\mathfrak{r}_{14}{}'\mathfrak{R}_0 + \mathfrak{r}_1{}'\mathfrak{R}_{16} + \mathfrak{r}_1{}'\mathfrak{R}_{15} + \mathfrak{r}_2{}'\mathfrak{R}_5 + \mathfrak{r}_3{}'\mathfrak{R}_6 + \mathfrak{r}_4{}'\mathfrak{R}_7 + \mathfrak{r}_5\,\mathfrak{R}_2\right.$$
$$\left. + \mathfrak{r}_6{}'\mathfrak{R}_3 + \mathfrak{r}_7{}'\mathfrak{R}_4 + \mathfrak{r}_{15}{}'\mathfrak{R}_1 + \mathfrak{r}_{16}{}'\mathfrak{R}_1\right\} e\,e_{\prime} \sin(x-z) \qquad [14]$$

$$+\frac{1}{(2-m-c)}\left\{2\,\mathfrak{r}_{15}{}' + 2\,\mathfrak{r}_0{}'\mathfrak{R}_{15} + 2\,\mathfrak{r}_{15}{}'\mathfrak{R}_0 + \mathfrak{r}_1{}'\mathfrak{R}_{14} + \mathfrak{r}_2{}'\mathfrak{R}_7 + \mathfrak{r}_3{}'\mathfrak{R}_5 + \mathfrak{r}_5{}'\mathfrak{R}_3 + \mathfrak{r}_7{}'\mathfrak{R}_2\right.$$
$$\left. + \mathfrak{r}_{14}{}'\mathfrak{R}_1\right\} e\,e_{\prime} \sin(2t-x+z) \qquad [15]$$

$$+\frac{1}{(2-3m+c)}\left\{2\,\mathfrak{r}_{16}{}' + 2\,\mathfrak{r}_0{}'\mathfrak{R}_{16} + 2\,\mathfrak{r}_{16}{}'\mathfrak{R}_0 + \mathfrak{r}_1{}'\mathfrak{R}_{14} + \mathfrak{r}_2{}'\mathfrak{R}_6 + \mathfrak{r}_4{}'\mathfrak{R}_5 + \mathfrak{r}_5{}'\mathfrak{R}_4 + \mathfrak{r}_6{}'\mathfrak{R}_2\right.$$
$$\left. + \mathfrak{r}_{14}{}'\mathfrak{R}_1\right\} e\,e_{\prime} \sin(2t+x+z) \qquad [16]$$

$$+\frac{1}{2m}\left\{2\,\mathfrak{r}_{17}{}' + 2\,\mathfrak{r}_0{}'\mathfrak{R}_{17} + 2\,\mathfrak{r}_{17}{}'\mathfrak{R}_0 + \mathfrak{r}_1{}'\mathfrak{R}_{19} + \mathfrak{r}_1{}'\mathfrak{R}_{18} + \mathfrak{r}_5{}'\mathfrak{R}_5 + \mathfrak{r}_6{}'\mathfrak{R}_7 + \mathfrak{r}_7{}'\mathfrak{R}_6 + \mathfrak{r}_{18}{}'\mathfrak{R}_1\right.$$
$$\left. + \mathfrak{r}_{19}{}'\mathfrak{R}_1\right\} e_{\prime}{}^2 \sin 2z \qquad [17]$$

$$+\frac{1}{(2-4m)}\left\{2\,\mathfrak{r}_{18}{}' + 2\,\mathfrak{r}_0{}'\mathfrak{R}_{18} + 2\,\mathfrak{r}_{18}{}'\mathfrak{R}_0 + \mathfrak{r}_1{}'\mathfrak{R}_{17} + \mathfrak{r}_5{}'\mathfrak{R}_6 + \mathfrak{r}_6{}'\mathfrak{R}_5 + \mathfrak{r}_{17}{}'\mathfrak{R}_1\right\} e_{\prime}{}^2 \sin(2t-2z) \qquad [18]$$

$$+\frac{1}{2}\left\{2\,\mathfrak{r}_{19}{}' + 2\,\mathfrak{r}_0{}'\mathfrak{R}_{19} + 2\,\mathfrak{r}_{19}{}'\mathfrak{R}_0 + \mathfrak{r}_1{}'\mathfrak{R}_{17} + \mathfrak{r}_5{}'\mathfrak{R}_7 + \mathfrak{r}_7{}'\mathfrak{R}_5 + \mathfrak{r}_{17}{}'\mathfrak{R}_1\right\} e_{\prime}{}^2 \sin(2t+2z) \qquad [19]$$

$$+\frac{1}{1-m}\left\{2\,\mathfrak{r}_{101}{}' + 2\,\mathfrak{r}_0{}'\mathfrak{R}_{101} + 2\,\mathfrak{r}_{101}{}'\mathfrak{R}_0 + \mathfrak{r}_1{}'\mathfrak{R}_{101}{}' + e^2\mathfrak{r}_2{}'\mathfrak{R}_{102} + e^2\mathfrak{r}_2{}'\mathfrak{R}_{103} + e^2\mathfrak{r}_3\,\mathfrak{R}_{102}\right.$$
$$\left. + e^2\mathfrak{r}_4{}'\mathfrak{R}_{103} + e_{\prime}{}^2 r_5\,\mathfrak{R}_{104} + e_{\prime}{}^2 r_5\,\mathfrak{R}_{105} + r_{101}\,\mathfrak{R}_1\right\} \sin t \qquad [101]$$

These examples will serve for the present to show how the development may be obtained from Table II.

M. DAMOISEAU has given (Mém. sur la Théorie de la Lune, p. 348,) the expression for $a\,\delta\,\frac{1}{r}$ in terms of the true longitude. In order to obtain a comparison of his results with those which may be obtained by the preceding method, it is necessary to transform his expressions, which may be done by LAGRANGE's theorem, into series containing explicitly the mean longitude.

If we suppose

$$\frac{a}{r} = A_0 + A_1 \cos(2\lambda' - 2m\lambda') + e A_2 \cos(c\lambda' - \varpi) + e A_3 \cos(2\lambda' - 2m\lambda' - c\lambda' + \varpi) + \&c.$$

$$s = B_{146}\gamma \sin(g\lambda' - \nu) + B_{147}\gamma \sin(2\lambda' - 2m\lambda - g\lambda' + \nu) + B_{148}\gamma \sin(2\lambda' - 2m\lambda' + g\lambda' - \nu) + \&c.$$

$$nt = \lambda' + C_1 \sin(2\lambda' - 2m\lambda') + e C_2 \sin(c\lambda' - \varpi) + e C_3 \sin(2\lambda - 2m\lambda' - c\lambda' + \varpi) + \&c.$$

in which expressions A, B, C are the same quantities as in M. DAMOISEAU's notation, the indices only being changed according to the remark, Phil. Trans. 1830, p. 246, in order that Table II. may be applicable to the transformation required; λ' is called v, and $\delta\,.\,\frac{1}{r'}$, δu in the notation of M. DAMOISEAU.

$$\frac{a}{r'} = A_0 + \frac{1}{2}(2-2m) A_1 C_1 + \frac{c}{2} e^2 A_2 C_2 + \frac{1}{2}(2-2m-c) e^2 A_3 C_3 + \frac{1}{2}(2-2m+c) e^2 A_4 C_4$$

$$+ \frac{m}{2} e_{\prime}^2 A_5 C_5 + \&c.$$

$$+ \Big\{ A_1 - \frac{1}{2} c e^2 A_2 C_3 + \frac{1}{2} c e^2 A_2 C_4 - \frac{1}{2}(2-2m-c) e^2 A_3 C_2 + \frac{1}{2}(2-2m+c) e^2 A_4 C_2$$

$$- \frac{1}{2} m e_{\prime}^2 A_5 C_6 + \frac{1}{2} m e_{\prime}^2 A_5 C_7 - \frac{1}{2}(2-3m) e_{\prime}^2 A_6 C_5 + \frac{1}{2}(2-m) e_{\prime}^2 A_7 C_5 \Big\} \cos 2t \qquad [1]$$

$$+ \Big\{ A_2 + \frac{1}{2}(2-2m) A_1 C_4 + \frac{1}{2}(2-2m) A_1 C_3 + \frac{1}{2}(2-2m-c) A_3 C_1$$

$$+ \frac{1}{2}(2-2m+c) A_4 C_1 \Big\} e \cos x \qquad [2]$$

$$+ \Big\{ A_3 + \frac{1}{2}(2-2m) A_1 C_2 + \frac{c}{2} A_2 C_1 \Big\} e \cos(2t - x) \qquad [3]$$

$$+ \Big\{ A_4 - \frac{1}{2}(2-2m) A_1 C_2 - \frac{c}{2} A_2 C_1 \Big\} e \cos(2t + x) \qquad [4]$$

$$+\left\{A_5+\frac{1}{2}(2-2m)A_1C_7+\frac{1}{2}(2-2m)A_1C_6+\frac{1}{2}(2-3m)A_6C_1\right.$$
$$\left.+\frac{1}{2}(2-m)A_7C_1\right\}e_{\prime}\cos z$$

[5]

$$+\left\{A_6+\frac{1}{2}(2-2m)A_1C_5+\frac{m}{2}A_5C_1\right\}e_{\prime}\cos(2t-z)$$

[6]

$$+\left\{A_7-\frac{1}{2}(2-2m)A_1C_5-\frac{m}{2}A_5C_1\right\}e_{\prime}\cos(2t+z)$$

[7]

$$+\left\{A_8+\frac{1}{2}(2-2m)A_1C_{10}+\frac{1}{2}(2-2m)A_1C_9-\frac{c}{2}A_2C_2+\frac{1}{2}(2-2m-c)A_3C_4\right.$$
$$\left.+\frac{1}{2}(2-2m+c)A_4C_3+\frac{1}{2}(2-2m-2c)A_9C_1+\frac{1}{2}(2-2m+2c)A_{10}C_1\right\}e^2\cos 2x$$

[8]

$$+\left\{A_9+\frac{1}{2}(2-2m)A_1C_8+\frac{c}{2}A_2C_3+\frac{1}{2}(2-2m-c)A_3C_2+cA_8C_1\right\}e^2\cos(2t-2x)$$

[9]

$$+\left\{A_{10}-\frac{1}{2}(2-2m)A_1C_8-\frac{c}{2}A_2C_4-\frac{1}{2}(2-2m+c)A_4C_2-cA_8C_1\right\}e^2\cos(2t+2x)$$

[10]

$$+\left\{A_{11}+\frac{1}{2}(2-2m)A_1C_{13}+\frac{1}{2}(2-2m)A_1C_{12}-\frac{c}{2}A_2C_5+\frac{1}{2}(2-2m-c)A_3C_7\right.$$
$$+\frac{1}{2}(2-2m+c)A_4C_6-\frac{m}{2}A_5C_2+\frac{1}{2}(2-3m)A_6C_4+\frac{1}{2}(2-m)A_7C_3$$
$$\left.+\frac{1}{2}(2-3m-c)A_{12}C_1\right\}ee_{\prime}\cos(x+z)$$

[11]

$$+\left\{A_{12}+\frac{1}{2}(2-2m)A_1C_{11}+\frac{c}{2}A_2C_6+\frac{1}{2}(2-2m-c)A_3C_5+\frac{m}{2}A_5C_3\right.$$
$$\left.+\frac{1}{2}(2-3m)A_6C_2+\frac{1}{2}(c+m)A_{11}C_1\right\}ee_{\prime}\cos(2t-x-z)$$

[12]

$$+\left\{A_{13}-\frac{1}{2}(2-2m)A_1C_{11}-\frac{c}{2}A_2C_7-\frac{1}{2}(2-2m+c)A_4C_5-\frac{m}{2}A_5C_4\right.$$
$$\left.-\frac{1}{2}(2-m)A_7C_2-\frac{1}{2}(c+m)A_{11}C_1\right\}ee_{\prime}\cos(2t+x+z)$$

[13]

$$+\left\{A_{14}+\frac{1}{2}(2-2m)A_1C_{16}+\frac{1}{2}(2-2m)A_1C_{15}+\frac{c}{2}A_2C_5+\frac{1}{2}(2-2m-c)A_3C_6\right.$$
$$+\frac{1}{2}(2-2m+c)A_4C_7+\frac{m}{2}A_5C_2+\frac{1}{2}(2-3m)A_6C_3+\frac{1}{2}(2-m)A_7C_4$$

$$+ \frac{1}{2}(2-3m+c) A_{15} C_1 + \frac{1}{2}(2-m-c) A_{16} C_1 \Big\} e e_, \cos(x-z)$$
[14]

$$+ \Big\{ A_{15} + \frac{1}{2}(2-2m) A_1 C_{14} + \frac{c}{2} A_2 C_7 - \frac{1}{2}(2-2m-c) A_3 C_5 - \frac{m}{2} A_5 C_3$$
$$+ \frac{1}{2}(2-m) A_7 C_2 + \frac{1}{2}(c-m) A_{14} C_1 \Big\} e e_, \cos(2t-x+z)$$
[15]

$$+ \Big\{ A_{16} - \frac{1}{2}(2-2m) A_1 C_{14} - \frac{c}{2} A_2 C_6 + \frac{1}{2}(2-2m+c) A_4 C_5 + \frac{m}{2} A_5 C_4$$
$$- \frac{1}{2}(2-3m) A_6 C_2 - \frac{1}{2}(c-m) A_{14} C_1 \Big\} e e_, \cos(2t+x-z)$$
[16]

$$+ \Big\{ A_{17} + \frac{1}{2}(2-2m) A_1 C_{19} + \frac{1}{2}(2-2m) A_1 C_{18} - \frac{m}{2} A_5 C_5 + \frac{1}{2}(2-3m) A_6 C_7$$
$$+ \frac{1}{2}(2-m) A_7 C_6 + \frac{1}{2}(2-4m) A_{18} C_1 + A_{19} C_1 \Big\} e_,^2 \cos 2z$$
[17]

$$+ \Big\{ A_{18} + \frac{1}{2}(2-2m) A_1 C_{17} + \frac{m}{2} A_5 C_6 + \frac{1}{2}(2-3m) A_6 C_5 + m A_{17} C_1 \Big\} e_,^2 \cos(2t-2z)$$
[18]

$$+ \Big\{ A_{19} - \frac{1}{2}(2-2m) A_1 C_{17} - \frac{m}{2} A_5 C_7 - \frac{1}{2}(2-m) A_6 C_5 - m A_{17} C_1 \Big\} e_,^2 \cos(2t+2z)$$
[19]

Similarly

$$s = \Big\{ B_{146} + \frac{1}{2}(2-2m+g) C_1 B_{148} - \frac{1}{2}(2-2m-g) C_1 B_{147} + \frac{1}{2}(c+g) e^2 C_2 B_{150}$$
$$- \frac{1}{2}(c-g) e^2 C_2 B_{149} + \frac{1}{2}(2-2m-c+g) e^2 C_3 B_{152} - \frac{1}{2}(2-2m-c-g) e^2 C_3 B_{151}$$
$$+ \frac{1}{2}(2-2m+c+g) C_4 B_{154} - \frac{1}{2}(2-2m+c-g) C_4 B_{153}$$
$$+ \frac{1}{2}(m+g) e_,^2 C_5 B_{156} - \frac{1}{2}(m-g) e_,^2 C_5 B_{155} \Big\} \gamma \sin y$$
[146]

$$+ \Big\{ B_{147} - \frac{g}{2} C_1 B_{146} - \frac{1}{2}(2-2m-c-g) e^2 C_2 B_{151} + \frac{1}{2}(2-2m+c-g) e^2 C_2 B_{153}$$
$$- \frac{1}{2}(c-g) e^2 C_3 B_{149} - \frac{1}{2}(c+g) C_4 B_{150} - \frac{1}{2}(2-3m-g) e_,^2 C_5 B_{157}$$
$$+ \frac{1}{2}(2-m-g) e_,^2 C_5 B_{159} \Big\} \gamma \sin(2t-y)$$
[147]

$$+\left\{B_{148}-\frac{g}{2}C_1B_{146}-\frac{1}{2}(2-2m-c+g)e^2C_2B_{152}+\frac{1}{2}(2-2m+c+g)e^2C_2B_{154}\right.$$
$$-\frac{1}{2}(c+g)e^2C_3B_{150}-\frac{1}{2}(c-g)C_4B_{149}-\frac{1}{2}(2-3m+g)e_{\prime}^2C_5B_{158}$$
$$\left.+\frac{1}{2}(2-m+g)e_{\prime}^2C_5B_{160}\right\}\gamma\sin(2t+y)$$
[148]

$$+\left\{B_{149}+\frac{1}{2}(2-2m+c-g)C_1B_{153}-\frac{1}{2}(2-2m-c+g)C_1B_{152}-\frac{g}{2}C_2B_{146}\right.$$
$$\left.+\frac{1}{2}(2-2m-g)C_3B_{147}-\frac{1}{2}(2-2m+g)C_4B_{148}\right\}e\gamma\sin(x-y)$$
[149]

$$+\left\{B_{150}+\frac{1}{2}(2-2m+c+g)C_1B_{154}-\frac{1}{2}(2-2m-c-g)C_1B_{151}-\frac{g}{2}C_2B_{146}\right.$$
$$\left.+\frac{1}{2}(2-2m+g)C_3B_{148}-\frac{1}{2}(2-2m-g)C_4B_{147}\right\}e\gamma\sin(x+y)$$
[150]

$$+\left\{B_{151}-\frac{1}{2}(c+g)C_1B_{150}+\frac{1}{2}(2-2m-g)C_2B_{147}-\frac{g}{2}C_3B_{146}\right\}e\gamma\sin(2t-x-y)$$
[151]

$$+\left\{B_{152}-\frac{1}{2}(c-g)C_1B_{149}+\frac{1}{2}(2-2m+g)C_2B_{148}-\frac{g}{2}C_3B_{146}\right\}e\gamma\sin(2t-x+y)$$
[152]

$$+\left\{B_{153}-\frac{1}{2}(c-g)C_1B_{149}-\frac{1}{2}(2-2m-g)C_2B_{147}-\frac{g}{2}C_4B_{146}\right\}e\gamma\sin(2t+x-y)$$
[153]

$$+\left\{B_{154}-\frac{1}{2}(c+g)C_1B_{150}-\frac{1}{2}(2-2m+g)C_2B_{148}-\frac{g}{2}C_4B_{146}\right\}e\gamma\sin(2t+x+y)$$
[154]

$$+\left\{B_{155}+\frac{1}{2}(2-m-g)C_1B_{159}-\frac{1}{2}(2-3m+g)C_1B_{158}-\frac{g}{2}C_5B_{146}\right\}e_{\prime}\gamma\sin(z-y)$$
[155]

$$+\left\{B_{156}+\frac{1}{2}(2-m+g)C_1B_{160}-\frac{1}{2}(2-3m-g)C_1B_{157}-\frac{g}{2}C_5B_{146}\right\}e_{\prime}\gamma\sin(z+y)$$
[156]

$$+\left\{B_{157}-\frac{1}{2}(m+g)C_1B_{156}+\frac{1}{2}(2-2m-g)C_5B_{147}\right\}e_{\prime}\gamma\sin(2t-z-y)$$
[157]

$$+\left\{B_{158}-\frac{1}{2}(m-g)C_1B_{155}+\frac{1}{2}(2-2m+g)C_5B_{148}\right\}e_{\prime}\gamma\sin(2t-z+y)$$
[158]

$$+\left\{B_{159}-\frac{1}{2}(m-g)C_1B_{155}-\frac{1}{2}(2-2m-g)C_5B_{147}\right\}e_{\prime}\gamma\sin(2t+z-y)$$
[159]

$$+\left\{B_{160}-\frac{1}{2}(m+g)\,C_1\,B_{156}-\frac{1}{2}\,(2-2m+g)\,C_5\,B_{148}\right\}e_{\prime}\gamma\sin(2t+z+y)$$

[160]

In order to verify these expressions, suppose

$$\frac{a}{r^{\backprime}}=A_2\,e\cos(c\,\lambda^{\backprime}-\varpi)\qquad s=\gamma\,B_{146}\sin(g\,\lambda^{\backprime}-\nu)\qquad n\,t=\lambda^{\backprime}+C_1\sin(2\,\lambda^{\backprime}-2\,m\,\lambda^{\backprime})$$

Then by Lagrange's theorem, neglecting A^3, $A^2\,C$, &c.

$$\frac{a}{r}=A_2\,e\cos x+c\,e\,A_2\,C_1\sin 2\,t\sin x \quad \text{nearly}$$

$$=A_2\,e\cos x+\frac{c\,A_2\,C_1}{2}\,e\cos(2\,t-x)-\frac{c\,A_2\,C_1}{2}\,e\cos(2\,t+x)$$

[2] [3] [4]

which terms are found in the expression which I have given above.

Again, by Lagrange's theorem,

$$s=\gamma\,B_{146}\sin y-g\,\gamma\,C_1\,B_{146}\sin 2\,t\cos y$$

$$=\gamma\,B_{146}\sin y-\frac{g\,C_1\,B_{146}}{2}\gamma\sin(2\,t-y)-\frac{g\,C_1\,B_{146}\,\gamma}{2}\sin(2\,t+y)$$

[146] [147] [148]

which terms are found in the expression which I have given above.

The numerical values of the quantities A, B, C, according to M. Damoiseau, are

	A_0	= ?	[30]	A_1	= ·00709538	[1]	A_2	= ?
*[31]	A_3	= ·2024622	[32]	A_4	= − ·00369361	[16]	A_5	= − ·0056375
[33]	A_6	= ·0289158	[34]	A_7	= − ·0030859	[2]	A_8	= ·003183 ?
[35]	A_9	= ·347942	[36]	A_{10}	= ·001970	[19]	A_{11}	= − ·19737
[41]	A_{12}	= ·516174	[42]	A_{13}	= ·0026238	[18]	A_{14}	= − ·286046
[39]	A_{15}	= − ·060625	[40]	A_{16}	= − ·014546	[17]	A_{17}	= − ·006930
[43]	A_{18}	= ·08125						
[30]	C_1	= − ·009216	[1]	C_2	= − 2·0044055	[31]	C_3	= − ·4138664
[32]	C_4	= ·012939	[16]	C_5	= − ·194385	[33]	C_6	= − ·394172
[34]	C_7	= ·0038267	[2]	C_8	= ·745169	[35]	C_9	= − ·286413
[36]	C_{10}	= − ·012575	[19]	C_{11}	= ·365516	[41]	C_{12}	= − 1·08891
[42]	C_{13}	= − ·008551	[18]	C_{14}	= − ·607534	[39]	C_{15}	= ·11587
[40]	C_{16}	= ·055936	[17]	C_{17}	= ·12755	[43]	C_{18}	= − ·11432

* These are the indices of the arguments in M. Damoiseau's work.

[0] $B_{147} = \cdot 0284942$ [2] $B_{149} = - \cdot 019169$ [6] $B_{151} = - \cdot 020788$
[5] $B_{153} = \cdot 006113$ [8] $B_{155} = - \cdot 081170$ [11] $B_{157} = \cdot 071237$
[10] $B_{159} = - \cdot 0033394$

Having found the coefficients of $\frac{a}{r'}$, those of $\frac{a}{r}$ are easily determined.

$$\frac{a}{r} = \frac{a}{r'(1+s^2)} = \frac{a}{r'}\left\{1 - \frac{s^2}{2}\right\}$$

$$= \frac{a}{r'}\left\{1 - \frac{\gamma^2}{4} - \frac{\gamma^2}{4} s^2{}_{147} + \frac{\gamma^2}{2} s_{147} \cos 2t \right.$$

$$\left. + \frac{\gamma^2}{4} \cos 2y - \frac{\gamma^2}{2} s_{147} \cos(2t - 2y)\right\}$$

If the coefficients of $\frac{a}{r}$ be called r_n',

$$r_0 = \left\{1 - \frac{\gamma^2}{4} - \frac{\gamma^2}{4} s^2{}_{147}\right\} r_0 + \frac{\gamma^2}{4} s_{147} r_1'$$

$$r_1 = \left\{1 - \frac{\gamma^2}{4} - \frac{\gamma^2}{4} s^2{}_{147}\right\} r_1'$$

$$r_2 = \left\{1 - \frac{\gamma^2}{4} - \frac{\gamma^2}{4} s^2{}_{147}\right\} r_2' + \frac{\gamma^2}{4} s_{147} r_3' + \frac{\gamma^2}{4} s_{147} r_4$$

$$r_3 = \left\{1 - \frac{\gamma^2}{4} - \frac{\gamma^2}{4} s^2{}_{147}\right\} r_3' + \left(1 - \frac{e^2}{8}\right) \frac{\gamma^2}{4} s_{147}$$

$$r_4 = \left\{1 - \frac{\gamma^2}{4} - \frac{\gamma^2}{4} s^2{}_{147}\right\} r_4' + \left(1 - \frac{e^2}{8}\right) \frac{\gamma^2}{4} s_{147}$$

$$r_5 = \left\{1 - \frac{\gamma^2}{4} - \frac{\gamma^2}{4} s^2{}_{147}\right\} r_5'$$

$$r_6 = \left\{1 - \frac{\gamma^2}{4} - \frac{\gamma^2}{4} s^2{}_{147}\right\} r_6' + \frac{\gamma^2}{4} s_{147} r_5'$$

$$r_7 = \left\{1 - \frac{\gamma^2}{4} - \frac{\gamma^2}{4} s^2{}_{147}\right\} r_7' + \frac{\gamma^2}{4} s_{147} r_6'$$

If we suppose

$$\frac{a}{r} = 1 + r_0 + e(1+f)\cos\Big(n(1+k)t + \varepsilon - \varpi\Big) + e_/ f_/ \cos\Big(n(1+k_/)t + \varepsilon_/ - \varpi_/\Big)$$

$a < a_/$ we find

$$r_0 = \frac{m_/}{\mu}\left\{\frac{a^3}{2a_/^3} b_{3,0} - \frac{a^2}{2a_/^2} b_{3,1}\right\} \qquad k = \frac{m_/}{\mu}\left\{\frac{a^3}{a_/^3} b_{3,0} - \frac{5a^2}{4a_/^2} b_{3,1}\right\}$$

$$f_/\{(1+k_/)^2(1 - 3r_0) - 1\} = \frac{m_/ a^2}{2\mu a_/^2} b_{3,2}$$

If $\quad n\{1+2r_0\} = \mathrm{n}$ and $\mathrm{n}^2 = \frac{\mu}{\mathrm{a}^3} \qquad a = \mathrm{a}\left\{1 + \frac{4}{3} r_0\right\}$

If 2 e is the coefficient of $\sin\left(n(1+k)t + \varepsilon - \varpi\right)$ in the expression for the longitude,

$$e(1+f) = \mathrm{e}(1 + k - r_0)$$

$$\frac{\mathrm{a}}{r} = 1 - \frac{1}{3} r_0 + \mathrm{e}\left\{1 + k - \frac{7}{3} r_0\right\} \cos\left(n(1+k)t + \varepsilon - \varpi\right)$$

$$+ e_{\prime} f_{\prime} \cos\left(n(1+k_{\prime})t + \varepsilon - \varpi_{\prime}\right)$$

$$= 1 - \frac{m_{\prime} a^3}{6\mu a_{\prime}^3} b_{3,0} + \frac{m_{\prime} a^2}{6\mu a_{\prime}^2} b_{3,1}$$

$$+ \mathrm{e}\left\{1 - \frac{m_{\prime} a^3}{6\mu a_{\prime}^3} b_{3,0} - \frac{m_{\prime} a^2}{12\mu a_{\prime}^2} b_{3,1}\right\} \cos\left(\mathrm{n}\left(1 - \frac{m_{\prime} a^2}{4\mu a_{\prime}^2} b_{3,1}\right) t + \varepsilon - \varpi\right)$$

$$+ e_{\prime} f_{\prime} \cos\left(n(1+k_{\prime})t + \varepsilon - \varpi_{\prime}\right)$$

$$\frac{r}{\mathrm{a}} = 1 + \frac{1}{3} r_0 - \mathrm{e}\left\{1 + k - \frac{5}{3} r_0\right\} \cos\left(n(1+k)t + \varepsilon - \varpi\right)$$

$$- e_{\prime} f_{\prime} \cos\left(n(1+k_{\prime})t + \varepsilon - \varpi_{\prime}\right)$$

$$= 1 + \frac{m_{\prime} a^3}{6\mu a_{\prime}^3} b_{3,0} - \frac{m_{\prime} a^2}{6\mu a_{\prime}^2} b_{3,1}$$

$$- \mathrm{e}\left\{1 + \frac{m_{\prime} a^3}{6\mu a_{\prime}^3} b_{3,0} - \frac{5 m_{\prime} a^2}{12\mu a_{\prime}^2} b_{3,1}\right\} \cos\left(\mathrm{n}\left(1 - \frac{m_{\prime} a^2}{4\mu a_{\prime}^2} b_{3,1}\right) t + \varepsilon - \varpi\right)$$

$$- e_{\prime} f_{\prime} \cos\left(\mathrm{n}(1+k_{\prime})t + \varepsilon - \varpi_{\prime}\right)$$

If $a < a_{\prime}$ as before, and

$$\frac{a_{\prime}}{r_{\prime}} = 1 + r_{\prime 0} + e_{\prime}(1+f')\cos\left(n_{\prime}(1+k')t + \varepsilon_{\prime} - \varpi_{\prime}\right) + e f_{\prime}' \cos\left(n_{\prime}(1+k_{\prime}')t + \varepsilon_{\prime} - \varpi\right)$$

we find

$$r_{\prime 0} = \frac{m}{\mu}\left\{\frac{1}{2} b_{3,0} - \frac{a}{2a_{\prime}} b_{3,1}\right\} \qquad k_{\prime} = \frac{m}{\mu_{\prime}}\left\{b_{3,0} - \frac{5a}{4a_{\prime}} b_{3,1}\right\}$$

$$f_{\prime}'\left\{(1+k_{\prime}')^2(1 - 3r_{\prime 0}) - 1\right\} = \frac{m a}{2\mu_{\prime} a_{\prime}} b_{3,2}$$

If $\mathrm{n}_{\prime}\{1+2r_0\} = \mathrm{n}_{\prime}$ and $\mathrm{n}_{\prime}^2 = \frac{\mu}{\mathrm{a}_{\prime}^3}, \qquad a_{\prime} = \mathrm{a}_{\prime}\left\{1 + \frac{4}{3} r_{\prime 0}\right\}$

$$\frac{a_{,}}{r_{,}} = 1 - \frac{m}{6\mu_{,}} b_{3,0} + \frac{m\,a}{6\mu_{,}a_{,}} b_{3,1}$$

$$+ e_{,} \left\{1 + \frac{m}{6\mu_{,}} b_{3,0} - \frac{m\,a}{12\mu_{,}a_{,}} b_{3,1}\right\} \cos\left(n_{,}\left(1 - \frac{m\,a}{4\mu_{,}a_{,}} b_{3,1}\right) t + \varepsilon_{,} - \varpi_{,}\right)$$

$$+ e f_{,}' \cos\left(n_{,}(1 + k_{,}')\, t + \varepsilon_{,} - \varpi\right)$$

μ is the mass of the sun + the mass of the disturbed planet, which is not of course the same for both, but the difference may be neglected in the planetary theory.

Laplace determines the arbitrary quantity $f_{,}$, upon the hypothesis that the coefficient of the argument $\sin\left(n\,(1 + k)\,t + \varepsilon - \varpi_{,}\right)$ in the expression for the longitude equals zero. According to the received theory of the moon, the true longitude is expressed in a series of angles consisting of various combinations of the quantities t, x, y and z, and their multiples and no others; and in this theory the angle $t + z$ occupies the place of the argument $n\,t + \varepsilon - \varpi_{,}$, so that omitting ε which accompanies t,

$$\frac{a}{r} = 1 + r_0 + e\,(1 + f) \cos(c\,n\,t - \varpi) + e_{,} f_{,} \cos(n\,t - n_{,}t + c_{,}n_{,}t - \varpi_{,})$$

$$\frac{a_{,}}{r_{,}} = 1 + r_{1/0} + e_{,}\,(1 + f') \cos(c'\,n_{,}t - \varpi_{,}) + e f_{,}' \cos(n_{,}t - n\,t + c\,n\,t - \varpi)$$

$$c^{*} = 1 - \frac{m_{,}a^2}{4\mu\,a_{,}^2} b_{3,1} \qquad c_{,} = 1 - \frac{m\,a}{4\mu_{,}a_{,}} b_{3,1} = 1 \text{ nearly}$$

$$n_{,}(c_{,} - 1) = n\,k_{,} = 0 \text{ nearly}$$

$$f_{,} \left\{(1 + k_{,})^2 (1 - 3\,r_0) - 1\right\} = \frac{m_{,}a^2}{2\mu\,a_{,}^2} b_{3,2} = \frac{15\,m_{,}a^4}{8\mu\,a_{,}^4}$$

$$r_0 = -\frac{m_{,}a^3}{2\mu\,a_{,}^3} \qquad f_{,} = \frac{5\,a}{4\,a_{,}}$$

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* c and g are determined by quadratic equations,

$$c = \frac{\sqrt{\left\{1 + \frac{m_{,}}{\mu}\left\{\frac{a^3}{2\,a_{,}^3} b_{3,0} - \frac{a^2}{a_{,}^2} b_{3,1}\right\}\right\}}}{1 + \frac{m_{,}}{\mu}\left\{\frac{a^3}{2\,a_{,}^3} b_{3,0} - \frac{a^2}{2\,a_{,}^2} b_{3,1}\right\}} = 1 - \frac{m_{,}a^2}{4\mu\,a_{,}^2} b_{3,1} \quad \text{nearly.}$$

This gives for the coefficient of $\sin(t+z)$ in the expression for the longitude

$$+\left\{\frac{5\,a}{2\,a_{\prime}}-\frac{3\,m_{\prime}\,a^4}{8\,\mu\,a_{\prime}^4}\right\}e_{\prime}$$

which in sexagesimal seconds is $21''\cdot7$, according to M. Damoiseau it should be $17''\cdot56$.

Finally,

$$\frac{\mathrm{a}}{r}=1+\frac{m_{\prime}\,a^3}{6\,\mu\,a_{\prime}^3}+\mathrm{e}\left\{1-\frac{7\,m_{\prime}\,a^3}{12\,\mu\,a_{\prime}^3}\right\}\cos x+\frac{5\,a}{4\,a_{\prime}}\,e_{\prime}\cos(t+z)$$

$$\lambda=\mathrm{n}\,t+2\,\mathrm{e}\sin x+\left\{\frac{5\,a}{2\,a_{\prime}}-\frac{3\,m_{\prime}\,a^4}{8\,\mu\,a_{\prime}^4}\right\}e_{\prime}\sin(t+z)$$

Substituting for $b_{3,1}$, $b_{3,2}$ their values in series

$$b_{3,1}=\frac{3\,a}{a_{\prime}}+\frac{3\,.\,3\,.\,5\,a^3}{2\,.\,4\,a_{\prime}^3}+\&c. \qquad b_{3,2}=\frac{3\,.\,5\,a^2}{4\,a_{\prime}^2}+\frac{3\,.\,3\,.\,5\,.\,7\,a^4}{2\,.\,4\,.\,6\,a_{\prime}^4}+\&c.$$

$$\mathrm{c}=1-\frac{3\,m_{\prime}\,a^3}{4\,\mu\,a_{\prime}^3} \qquad \mathrm{c}_{\prime}=1-\frac{3\,m\,a^2}{4\,\mu_{\prime}a_{\prime}^2}$$

I have shown, Phil. Trans. 1832, p. 38, that when $a<a_{\prime}$

$$g=1+\frac{m_{\prime}}{\mu}\left\{\frac{a^3}{a_{\prime}^3}\,b_{3,0}-\frac{3\,a^2}{4\,a_{\prime}^3}\,b_{3,1}\right\}$$

$$g\,n=\mathrm{n}\left\{1+\frac{m_{\prime}\,a^2}{4\,\mu\,a_{\prime}^2}\,b_{3,1}\right\}$$

Similarly it may be shown that

$$g_{\prime}=1+\frac{m}{\mu_{\prime}}\left\{b_{3,0}-\frac{3\,a}{4\,a_{\prime}}\,b_{3,1}\right\}$$

$$g_{\prime}n_{\prime}=\mathrm{n}_{\prime}\left\{1+\frac{m\,a}{4\,\mu_{\prime}a_{\prime}}\,b_{3,1}\right\}$$

The arguments

$$n\,t-\nu,\ n\,t-\nu_{\prime},\ n\,t_{\prime}-\nu_{\prime}\ \text{and}\ n_{\prime}t-\nu$$

occupy the same place in the expression for the latitude as

$$n\,t-\varpi,\ n\,t-\varpi_{\prime},\ n_{\prime}t-\varpi_{\prime}\ \text{and}\ n_{\prime}t-\varpi$$

in the expression for the radius vector. Similar methods may be employed to determine the arbitrary quantities, so that no other angles occur in the expression for s except the quantities t, x, z, y, and if the quantities c and g are rational, no imaginary angles can be introduced.